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LETTER TO THE EDITOR

Extended scaling relations for the chiral and cubic crossover exponents

Marcel den Nijs

Department of Physics, University of Washington, Seattle, Washington 98195, USA

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Abstract. The critical dimensions $x_{CH}(q)$ of the chiral and $x_{CB}(q)$ of the cubic operator in the two-dimensional q-state Potts model satisfy the extended scaling relations $x_{CH} = (n^2-y^2)/(4x)+x$ and $x_{CB} = (m^2-y^2)/(4x)+x$ with $x+y = 2, 2\cos(\pi y/2) = \sqrt{q}$, n an odd and m an even integer; $n=1$ and $m=0$ give the leading exponents. At $q=3$ x_{CH} is relevant, but takes the special value $x_T+1 = 9/5$. The crossover exponent at the percolation point in the bond-diluted random Ising model, determined by x_{CB} at $q=1$, is equal to one. Along the Baxter line in the Ashkin–Teller, eight-vertex and ANNNI model $x_{CH} = x_T/4+1/x_T$.

The exact values of the chiral and cubic crossover exponents are obtained in the two dimensional Potts model and along the Baxter line in the eight-vertex, Ashkin–Teller, and ANNNI model. First it is explained why chiral symmetry breaking is considered, next the implications of the extended scaling relations are discussed, and finally their derivation is sketched.

An He monolayer adsorbed on graphite (Tejwani *et al* 1980), and an H monolayer adsorbed on Fe(110) (Imbihl *et al* 1982), are examples of experimental systems with phase transitions described by the three-state Potts model. The Bloch walls in the Potts model represent domain walls in the monolayer. Divide the substrate into cells, take the cell size similar to the domain wall width, and let Potts spins $\theta = 2\pi n/q$ represent the $q=3$ commensurate states of the monolayer in the cells. In the conventional Potts model all types of Bloch walls cost the same energy. In the monolayer however there will be an energy difference dE between anti-clockwise walls, $d\theta_s = \theta_{s+\hat{e}} - \theta_s = -2\pi/q$, and clockwise walls, $d\theta_s = 2\pi/q$. The chiral field dE couples to the chemical potential, i.e. (like the fugacity of vacancies) is conjugate to the monolayer density. The phase diagram contains four types of phase. Incommensurate fluids have a finite correlation length ξ, and a modulation in the correlation functions with pitch Q. In commensurate fluids, Q is commensurate with the substrate periodicity, and the diffraction peaks are locked-in at commensurate positions. The distinction becomes obscure when $\xi \simeq Q^{-1}$. The incommensurate floating solid phase also has an incommensurate pitch, but now because dislocations in the domain wall network are bound in pairs, and because meander (and/or breathing) fluctuations are massless, ξ is infinite. Commensurate solids have an order parameter at commensurate Q and a finite ξ.

Most transitions between these four phases are understood. The scaling properties at commensurate solid–fluid phase boundaries are known (for a review see Schick 1981). Melting of the IC-floating solid into the IC-fluid belongs to the same universality class as melting on a smooth substrate. The transition between a commensurate solid

and an IC-floating solid is a Pokrovski–Talapov (1979) transition for striped phases and a first-order transition (Villain 1980) for honeycomb domain wall networks. IC-fluids are separated from commensurate fluids by disorder lines (see e.g. Emery and Peschel 1981).

IC-fluids and IC-floating solids only differ in the shape of their diffraction peaks. The incommensurate phase reported by Jaubert *et al* (1981) in Xe adsorbed on copper probably is an IC-fluid, and the C–I transition must be a disorder line. At the disorder line the pitch vanishes with the same exponent $\beta = \frac{1}{2}$ as at a Pokrovski–Talapov transition (den Nijs 1984).

However, the nature of a direct phase boundary between an IC-fluid and C-solid is not yet resolved. Such transition is absent in the limit where one type of domain wall dominates ($dE \gg 0$), and the system can be described by a fermion quantum field theory (for a review see den Nijs 1984). Numerical evidence (Howes *et al* 1983, Howes 1983) for the chiral three-state Potts model (Ostlund 1981) indicates that the Kosterlitz–Thouless and Pokrovski–Talapov transition line for large dE merge at a Lifshitz point and that a single phase transition line remains for $0 < dE < dE_L$. In the self-dual version of the chiral three-state Potts model, the Lifshitz point is located at $dE_L = 2$, and also its scaling properties are known exactly. Its thermodynamic exponents are identical to those at $dE = 0$, while scaling is anisotropic; $\xi_\parallel$ along the walls diverges with an Ising exponent $\nu_\parallel = 1$ (Howes *et al* 1983). The argument by Bohr *et al* (1984) for the absence of a Lifshitz point actually only implies that $dE_L \to 0$ in a system where the energy of dislocations goes to infinity. Within the numerical accuracy the exponents along the critical line $0 < dE < dE_L$ are the same as at $dE = 0$ (Howes *et al* 1983, Howes 1983). The behaviour of the pitch at the fluid side is not resolved. The two possibilities are. (a) This is a commensurate melting line, the fluid is commensurate and only at higher temperatures becomes incommensurate via a disorder line. (b) This is a direct phase boundary between the commensurate solid and the incommensurate fluid, and the critical exponent x_Q for the pitch $Q \sim \xi^{-xQ}$ has to be determined. Huse and Fisher (1982) expect for uniaxial systems $x_Q = 1$.

The value of the chiral crossover exponent obtained here, determines the stability of the model at $dE = 0$ with respect to chirality. Consider the chiral pair correlation function in the Potts model at $dE = 0$,

$$G_{CH}(r) = 2\langle \delta_{d\theta_s, 2\pi/q} \delta_{d\theta_{s+r}, 2\pi/q} \rangle - \frac{2}{(q-1)^2} G_{DD}(r) - \frac{q-3}{q-1} G_{CB}(r) \tag{1}$$

with $G_{DD}(r) = \langle (1 - \delta_{d\theta_s,0})(1 - \delta_{d\theta_{s+r},0}) \rangle$ the energy–energy correlation function, and $G_{CB}(r)$ the cubic correlation function defined in (4). At criticality $G_{CH}(r)$ decays as a powerlaw $r^{-2x_{CH}}$. As shown below x_{CH} satisfies the extended scaling relation

$$x_{CH} = (n^2 - y^2)/(4x) + x \tag{2}$$

for $q \leqslant 4$, with $x = 2 - y$, $2\cos(\pi y/2) = \sqrt{q}$, and n an odd integer; $n = 1$ gives the leading exponent. (2) applies both to uniaxial and isotropic chirality. The cubic component in (1) is sub-dominant except at $q = 4$ (see below). The chiral operator O_{CH} is irrelevant along the entire tri-critical branch ($-1 \leqslant y < 0$), and vanishes at $q = 2$ critical points. So for small $dE \neq 0$ the model flows under scaling to $dE = 0$; the phase boundary remains a commensurate melting transition (possibility *a*).

At $q = 3$ critical points O_{CH} is relevant, $x_{CH} = \frac{9}{5}$; both possibilities (a) and (b) are still allowed. However, x becomes equal to the critical dimension $x_T + 1 = \frac{9}{5}$ of the gradient of the energy operator, dO_D/ds, and also x_V associated to vacancies does not

introduce a new independent exponent, $x_V = x_T + 2$ (Nienhuis 1982). A similar lock-in of exponents takes place along the Baxter line at the point where the eight-vertex model decouples into two Ising models (see e.g. Kadanoff and Brown 1979). There the singularities can be described by a reduced set of operators. So one might speculate that along the entire critical line $0 < dE < dE_L$ the scaling behaviour is the same as at $dE = 0$ except for an increasing dO_D/ds component. The latter might introduce the pitch at the fluid side. This would imply $x_Q \geqslant \frac{9}{5}$, i.e. that the correlation length diverges slower than the inverse pitch. Recall that, in the IC-floating solid, the pitch Q, the displacement of the diffraction peaks Q_{DF}, and the surplus of clockwise walls $Q_{CH} = \langle O_{CH} \rangle$ all scale as a length, $x_Q = x_{DF} = x_{CH} = 1$; some walls form droplets which do not contribute to the modulation ($Q_{CH} > Q$), but do not affect the scaling behaviour because free dislocations are absent. Along the $0 < dE < dE_L$ critical line only the inequality $x_Q \geqslant x_{CH}$ remains, because it might be that droplets form at all length scales. $x_Q \geqslant \frac{9}{5}$ implies the presence of a disorder line like region in the fluid where $\xi \simeq Q^{-1}$, and confirms that the exponents have the $dE = 0$ values; in the scaling limit no modulation remains within the correlation length. This also agrees with the hard hexagon model where $x_Q = 2$ and the exponents have the $dE = 0$ three-state Potts values (Baxter and Pearce 1982, Huse 1984). In Kr adsorbed on graphite, the misfit of the diffraction peaks vanishes as $|\Delta\mu|^{1/3}$ (Chinn and Fain 1977); this may indicate $x_Q = \frac{2}{5}$, but more recent experiments (Stephens *et al* 1983) suggest to me a disorder line, or $x_Q > 1$ with a disorder line like cross-over region in the fluid where $\xi \simeq Q^{-1}$.

Along the Baxter line in the eight-vertex, Ashkin–Teller and ANNNI model, x_{CH} satisfies the extended scaling relation

$$x_{CH} = x_T/4 + 1/x_T. \tag{3}$$

(3) agrees with (2) at the point where the Baxter line intersects the four-state Potts model ($x_T = 1/x$).

In cubic symmetry breaking the q states are divided in n groups each containing $m = q/n$ states. Bloch walls between domains of the same group and different groups get different energy. As shown below the critical dimension x_{CB} in the cubic pair correlation function

$$G_{CB}(r) = \frac{n^2(q-1)}{q(q-n)(n-1)}(1-\delta_{dn_s,0})(1-\delta_{dn_{s+r},0}) - \frac{q(n-1)}{(q-n)(q-1)} G_{DD}(r) \tag{4}$$

satisfies the extended scaling relation

$$x_{CB} = (m^2 - y^2)/(4x) + x \tag{5}$$

with m an even integer and $m = 0$ the leading exponent. O_{CB} is irrelevant at $q < 4$. At $q = 4$, O_{CB} is marginal and generates the Baxter line in the Ashkin–Teller model (see e.g. Kohmoto *et al* 1981). Since O_{CH} includes a cubic component, see (1), there will also be a Baxter line in the chiral four-state Potts model. The exponents will vary continuously with dE until $x_T = 4 - 2\sqrt{3}$, where (see (3)) x_{CH} becomes relevant.

Finally a quite different application: the crossover at the percolation point in the bond-diluted random Ising model is governed by the cubic operator at $q = 1$ (see e.g. Domany 1978). The crossover exponent is equal to one, because $x_{CB}(1) = x_T(1) = \frac{5}{4}$.

Now follows a brief sketch of the derivation of equations (2), (3) and (5). At $dE = 0$ the Potts model can be reformulated as a six-vertex Coulomb gas. Earlier, the energy (Black and Emery 1981), magnetic (den Nijs 1983) and vacancy-umklapp

(Nienhuis 1982, den Nijs 1981) excitations have been identified with spin-wave charges $Q=(S, 0)$. At criticality the Coulomb gas consists of two condensator plates each with a surface charge $Q=(y, 0)$ and a charge $Q=(-2y, 0)$ inside. Equation (20) in den Nijs (1983), modified below for the condensator-type boundary condition, shows that the energy of this state is identical to that of an empty condensator without surface charges. This is a remarkable property of the short-range part of the interactions. The expressions for G_{CB} and G_{CH} derived below, (6) and (7), are similar to (27) in den Nijs (1983) for the spin–spin correlation function G_H. $G_H(r)$, $G_{CB}(r)$, and $G_{CH}(r)$ are the probabilities that the $Q=(-2y, 0)$ bulk charge splits over a distance r into a $Q=(n-y, 0)$, $Q=(-n-y, 0)$ pair, a $Q=(m-y, 1)$, $Q=(-m-y, -1)$ pair, and a $Q=(n-y, 1)$, $Q=(-n-y, -1)$ pair respectively (n = odd, m = even). $Q=(0, V)$ represents a vortex charge.

At short distances the interactions in the Gaussian and the six-vertex Coulomb gas differ, but at large distances r, a pair $Q=(S_1, V_1)$, $Q=(S_2, V_2)$ contributes $f_{12}(r) \simeq 2[S_1S_2/(4x)+V_1V_2x]\log(r)$ to the free energy (see e.g. Kadanoff and Brown 1979); $x=2-y$ parametrises the critical line in the Potts model and plays the role of temperature in the six-vertex model. The exact solution of the Potts model at criticality (Baxter *et al* 1976) gives the relation between x and the Gaussian inverse temperature $K=x/\pi$. This, together with the identification of the charges, implies the extended scaling relations (2) and (5).

For details of notation and method see den Nijs (1983). First $G_{DD}G_{CB}$ and G_{CH} are rewritten in the random cluster model representation as expectation values of weight functions W_α. W_α is zero if $\boldsymbol{s}$ and $\boldsymbol{s}+\hat{\boldsymbol{e}}$ and/or $\boldsymbol{s}+\boldsymbol{r}$ and $\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}$ belong to the same cluster. Four types of graphs contribute: DD (all four sites belong to different clusters), DC (only $\boldsymbol{s}$ or $\boldsymbol{s}+\hat{\boldsymbol{e}}$ is connected with $\boldsymbol{s}+\boldsymbol{r}$ or $\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}$), CC1 ($\boldsymbol{s}$ is connected with $\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}$, and $\boldsymbol{s}+\hat{\boldsymbol{e}}$ with $\boldsymbol{s}+\boldsymbol{r}$), and CC2 ($\boldsymbol{s}$ is connected with $\boldsymbol{s}+\boldsymbol{r}$, and $\boldsymbol{s}+\hat{\boldsymbol{e}}$ with $\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}$). $W_\alpha=[\mathrm{DD}, \mathrm{DC}, \mathrm{CC1}, \mathrm{CC2}]$ take the values: $W_{DD}=[(1-1/q)^2, (1-1/q)^2, 1-1/q, 1-1/q]$, $W_{CB}=[0, 0, 1/q, 1/q]$, and $W_{CH}=[0, 0, 1/q, -1/q]$. W_{DD} is equivalent to $\tilde{W}_{DD}$, with $\tilde{W}_{DD}=1$ if the bond between $\boldsymbol{s}$ and $\boldsymbol{s}+\hat{\boldsymbol{e}}$ and between $\boldsymbol{s}+\boldsymbol{r}$ and $\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}$ are both absent, and $\tilde{W}_{DD}=0$ otherwise.

The six-vertex representations for W_{CB} and W_{CH} are similar to the one for G_H in equation (21) of den Nijs (1983)

$$G_{CB}(r)=\left\langle \exp\left(-i\pi y \sum_{\text{path},\mathbf{0}}^{\boldsymbol{s}} \sigma\right) O_{0,-1}(R_{\boldsymbol{s}})O_{0,1}(R_{\boldsymbol{s}+\boldsymbol{r}}) \exp\left(-i\pi y \sum_{\text{path},\mathbf{0}}^{\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}} \sigma\right)\right\rangle, \tag{6}$$

$$G_{CH}(r)=\left\langle \exp\left(i\pi(-1-y) \sum_{\text{path},\mathbf{0}}^{\boldsymbol{s}} \sigma\right) O_{0,-1}(R_{\boldsymbol{s}})O_{0,1}(R_{\boldsymbol{s}+\boldsymbol{r}}) \times \exp\left(i\pi(1-y) \sum_{\text{path},0}^{\boldsymbol{s}+\boldsymbol{r}+\hat{\boldsymbol{e}}} \sigma\right)\right\rangle. \tag{7}$$

The paths intersect the arrows at the polygons; $\sigma=\sigma(R, R')=\pm 1$, with vertex R (R') to the right (left) of the path, represents the direction of the arrow. $O_{0,\pm 1}$, with charge $Q=(0, \pm 1)$, are energy excitations of the eight-vertex model (Kadanoff and Brown 1979). In (6) and (7) they restrict the graphs to type CC1 and CC2; the arrows are reversed along segments of two polygons, such that vertex $R_{\boldsymbol{s}}$ between $\boldsymbol{s}$ and $\boldsymbol{s}+\hat{\boldsymbol{e}}$ becomes a sink and $R_{\boldsymbol{s}+\boldsymbol{r}}$ a source of four arrows. The two string operators in (6) compensate, if polygons surrounding site $\mathbf{0}$ are excluded, for the error counting factors $\sqrt{q}$ introduced by this. Take $\mathbf{0}$ at the lattice boundary, and assume that along the

boundary the interactions are infinitely strong. Now the charges are located inside a box of constant potential. The string operators represent charges $Q=(-y,0)$ at site s and $s+r+\hat{e}$, and a surface charge $Q=(2y,0)$. In (7) a charge $Q=(-1,0)$ is added at site s and $Q=(1,0)$ at $s+r+\hat{e}$, because W_{CH} is equal to 1 (-1) if s and $s+r+\hat{e}$ do (not) belong to the same cluster. The charges are unique modulo $Q=(2,0)$ (see also den Nijs (1983)). The higher-order representations of the charges are associated to the subdominant exponents.

The box can be replaced by a condensator, which allows for groundstates with domain walls parallel to the plates, and is also useful in transfer matrix studies. Consider a cylinder of length T. Assume that at $t=0$ and $t=T$ the interactions between the Potts spins are infinitely strong. The phase factors fail to count factors $\sqrt{q}$ for open polygons, wrapped around the cylinder. Choose a site **0** at $t=0$, a site **1** at $t=T$, and a path between them. The Potts model translates into the same six-vertex Coulomb gas as before, but subject to the new boundary condition, and the constraint that a charge $Q=(-y,0)$ is located at **0** and a charge $Q=(y,0)$ at **1**. Since no polygon surrounds site **0**, (20) in den Nijs (1983) remains valid. At criticality the condensator contains a single point charge $Q=(-2y)$ and a surface charge $Q=(y,0)$ at both plates.

The chiral operator in the Ashkin–Teller model has been reformulated into the eight-vertex language (Kohmoto *et al* 1981) by Schultz (1984). O_{CH} should be identified with a charge $Q=(-1,1)$, which gives (3). In the very anisotropic limit the ANNNI model can be mapped into an eight-vertex model (see e.g. Emery and Peschel 1981). The critical point where the model decouples into two Ising models, generalises into the Baxter line, when the core energy of dislocations is varied. Along the Baxter line $O_{CH}=O_s^{8V}$ has a charge $Q=(2,\frac{1}{2})$ (tables I–II in den Nijs 1981) which leads again to (3).

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